

REIDEMEISTER-SCHREIER'S ALGORITHM FOR 2-COVERINGS OF SEIFERT MANIFOLDS

A. BAUVAL, C. HAYAT

ABSTRACT. It is classical that given any Seifert structure on N , Reidemeister-Schreier's algorithm produces a presentation of all index 2 subgroups of $\pi_1(N)$, described as the fundamental group of some Seifert manifolds. The new result of this article is concise formulas that gather all possible cases.

1. INTRODUCTION

The index 2 subgroups of $\pi_1(N)$ are kernel of epimorphisms $\varphi : \pi_1(N) \rightarrow \mathbb{Z}_2$. When N is a Seifert manifold (described by its Seifert invariants) and one needs a list of all its 2-coverings, it is necessary to explicit a combinatoric way to gather together all the data. Theorems 1 and 3 give Reidemeister-Schreier concise answers, [2], [6].

The notations are given in the section after this introduction. Section 3 studies the situation where the morphism φ maps the generator corresponding to the regular fiber to 1. This is Theorem 1. Theorem 3 stating the result when φ maps the generator corresponding to the regular fiber to 0 is expressed in the fourth section. The following subsections prove this theorem. In the first subsection, a crucial lemma (Lemma 10) proves that if two morphisms from $\pi_1(N)$ to \mathbb{Z}_2 map $m \geq 0$ exceptional fibres to 1 and all the other generators to 0 then their kernels are isomorphic. This gives importance to Theorem 11 which explicits the kernel with these hypothesis. The second subsection studies the situation where φ maps some generators corresponding to the basis to 1 and all the other generators to 0. Theorem 3 is proved.

Each index 2 subgroup of $\pi_1(N)$ is the fundamental group of a Seifert manifold M and an associated free involution τ . The motivation of this study came to us through the study of Borsuk-Ulam type theorem for (M, τ) [3], [1].

2. SEIFERT INVARIANTS FOR THE KERNEL

N is any Seifert manifold (orientable or not), as introduced in [5].

Following the notations of [4], from now on, N will be a Seifert manifold described by a list of Seifert invariants

$$\{e; (\epsilon, g); (a_1, b_1), \dots, (a_n, b_n)\}.$$

We do not need them to be “normalized” (as defined in [5] or [4]): we only assume that e is an integer, the type ϵ is detailed below, g is the genus of the base surface, and for each k , the integers a_k, b_k are coprime and $a_k \neq 0$.

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Such invariants give the following presentation of the fundamental group of N :

$$\pi_1(N) = \left\langle \begin{array}{c} s_1, \dots, s_n \\ v_1, \dots, v_{g'} \\ h \end{array} \middle| \begin{array}{l} [s_k, h] \text{ and } s_k^{a_k} h^{b_k}, \quad 1 \leq k \leq n \\ v_j h v_j^{-1} h^{-\varepsilon_j}, \quad 1 \leq j \leq g' \\ h^{-e} s_1 \dots s_n V \end{array} \right\rangle.$$

- The type ϵ of N equals:
 - o_1 if both the base surface and the total space are orientable (which forces all ε_j 's to equal 1);
 - o_2 if the base is orientable and the total space is non-orientable, hence $g \geq 1$ (all ε_j 's are assumed to equal -1);
 - n_1 if both the base and the total space are non-orientable (hence $g \geq 1$) and moreover, all ε_j 's equal 1;
 - n_2 if the base is non-orientable (hence $g \geq 1$) and the total space is orientable (which forces all ε_j 's to equal -1);
 - n_3 if both the base and the total space are non-orientable and moreover, all ε_j 's equal -1 except $\varepsilon_1 = 1$, and $g \geq 2$;
 - n_4 if both the base and the total space are non-orientable and moreover, all ε_j 's equal -1 except $\varepsilon_1 = \varepsilon_2 = 1$, and $g \geq 3$.
- The orientability of the base and its genus g determine the number g' of the generators v_j 's and the word V in the last relator of $\pi_1(N)$:
 - when the base is orientable, $g' = 2g$ and $V = [v_1, v_2] \dots [v_{2g-1}, v_{2g}]$;
 - when the base is non-orientable, $g' = g$ and $V = v_1^2 \dots v_g^2$.
- The generator h corresponds to the generic regular fibre.
- The generators s_k for $1 \leq k \leq n$ correspond to (possibly) exceptional fibres.

The subgroups of index 2 of $\pi_1(N)$ are the kernel of epimorphism $\varphi : \pi_1(N) \rightarrow \mathbb{Z}_2$. The two next subsections describe $\text{Ker}(\varphi)$ as the fundamental group of some Seifert manifold given by a similar list of invariants, when $\varphi(h) = 1$ (Theorem 1) and when $\varphi(h) = 0$ (Theorem 3).

3. IF φ MAPS h TO 1

Theorem 1. *If φ maps h to 1 then its kernel is the fundamental group of the Seifert manifold given by the following invariants:*

$$\left\{ \frac{e-m}{2} - m'; (\epsilon, g); (a_1, b'_1), \dots, (a_n, b'_n) \right\},$$

where $b'_k = \begin{cases} \frac{b_k}{2} & \text{if } b_k \text{ is even} \\ \frac{a_k + b_k}{2} & \text{if } b_k \text{ is odd} \end{cases}$, m is the number of odd b_k 's, and

$$\begin{cases} m' = 0 & \text{if } \epsilon = o_1, n_2 \\ m' \equiv \sum \varphi(v_j) \pmod{2} & \text{if } \epsilon = o_2, n_1 \\ m' \equiv \varphi(v_1) \pmod{2} & \text{if } \epsilon = n_3 \\ m' \equiv \varphi(v_1) + \varphi(v_2) \pmod{2} & \text{if } \epsilon = n_4. \end{cases}$$

Note that in the non-orientable cases, m' is only determined modulo 2, which is sufficient to determine the Seifert manifold.

Proof. Necessarily, all a_k 's are odd, $\varphi(s_k) = b_k \pmod{2}$, and $e + m$ is even. Let us choose a presentation of $\pi_1(N)$ adapted to φ by keeping h untouched but taking

new generators s'_k, v'_j mapped to 0 by φ :

$$s'_k = \begin{cases} s_k & \text{if } b_k \text{ is even} \\ h^{-1}s_k & \text{if } b_k \text{ is odd} \end{cases} \quad v'_k = \begin{cases} v_k & \text{if } \varphi(v_k) = 0 \\ h^{-1}v_k & \text{if } \varphi(v_k) = 1. \end{cases}$$

The new presentation of $\pi_1(N)$ corresponds to the Seifert invariants

$$\{e - m - 2m'; (\epsilon, g); (a_1, 2b'_1), \dots, (a_n, 2b'_n)\},$$

where the b'_k 's and m are as stated, and

$$m' = \begin{cases} 0 & \text{if } \epsilon = o_1, n_2 \\ \#\{j \text{ odd} \mid \varphi(v_j) = 1\} - \#\{j \text{ even} \mid \varphi(v_j) = 1\} & \text{if } \epsilon = o_2 \\ \#\{j \mid \varphi(v_j) = 1 \text{ and } \varepsilon_j = 1\} & \text{if } \epsilon = n_1, n_2, n_3, n_4 \end{cases}$$

(hence m' fullfills the condition of the statement).

Choosing $q = h$, Reidemeister-Schreier's algorithm produces a presentation of $\text{Ker}(\varphi)$ with

- generators:
 - for $1 \leq k \leq n$, $(y_k, y'_k) = (s'_k, qs'_kq^{-1})$
 - for $1 \leq j \leq g'$, $(x_j, x'_j) = (v'_j, qv'_jq^{-1})$
 - $(z, z') = (hq^{-1}, qh)$
- relations:
 - $z = 1$
 - for $1 \leq k \leq n$, $y'_k = y_k$, $[y_k, z'] = 1$ and $y_k^{a_k} z'^{b'_k} = 1$
 - for $1 \leq j \leq g'$, $x_j z' x_j^{-1} z'^{-\varepsilon_j} = 1$ and $x'_j = \begin{cases} x_j & \text{if } \varepsilon_j = 1 \\ z' x_j & \text{if } \varepsilon_j = -1 \end{cases}$
 - $y_1 \dots y_n W = z'^{(e+m+2m')/2}$, with

$$W = \begin{cases} [x_1, x_2] \dots [x_{2g-1}, x_{2g}] & \text{if } \epsilon = o_1, o_2 \\ x_1^2 \dots x_g^2 & \text{if } \epsilon = n_1, n_2, n_3, n_4. \end{cases}$$

Eliminating the redundant generators z, y'_k, x'_k yields the result. \square

4. IF φ MAPS h TO 0

4.0.1. Results of the two next subsections. Denote by m the number of s_k 's mapped to 1 by φ and assume (if $m > 0$) that these m s_k 's are the first ones. (This reordering of the s_k 's may be achieved by an obvious change of presentation of $\pi_1(N)$, using repeatedly the equation $ss' = (ss's^{-1})s$.) The next theorem announces the conclusion of Theorems 11 and 15, which will be proved in the two next subsections. The following notations will be used to state the results.

Notation 2. The notations F_{OC} and F_m will respectively denote

$$F_{OC} = (a_1, b_1), (a_1, -b_1), (a_2, b_2), (a_2, -b_2), \dots, (a_n, b_n), (a_n, -b_n)$$

$$F_m = (a_1/2, b_1), (a_2/2, b_2), \dots, (a_m/2, b_m),$$

$$(a_{m+1}, b_{m+1}), (a_{m+1}, b_{m+1}), (a_{m+2}, b_{m+2}), (a_{m+2}, b_{m+2}), \dots, (a_n, b_n), (a_n, b_n)$$

Theorem 3. If $\varphi(h) = 0$, denoting by m the number of s_k 's mapped to 1 by φ and assuming these are the first ones, $\text{Ker}(\varphi)$ is the fundamental group of the Seifert manifold given by the following invariants:

- (Orientation covers) If $m = 0$ and

- if $\epsilon = o_2$ and φ maps all v_j 's to 1: $\{0; (o_1, 2g - 1); F_{OC}\}$
- if $\epsilon = n_1$ and φ maps all v_j 's to 1: $\{0; (o_1, g - 1); F_{OC}\}$
- if $\epsilon = n_3$ and φ sends only v_1 to 1, or if $\epsilon = n_4$ and φ sends only v_1, v_2 to 1: $\{0; (n_2, 2g - 2); F_{OC}\}$
- (Exotic cases) If $m = 0$ and $\epsilon = n_2, n_3, n_4$ and φ maps all v_j 's to 1:
 - if $\epsilon = n_2$: $\{2e; (o_1, g - 1); F_0\}$
 - if $\epsilon = n_3, n_4$: $\{0; (o_2, g - 1); F_0\}$
- (Ordinary cases) In all other cases: $\{e'; (\epsilon', G); F_m\}$ with

$$e' = \begin{cases} 2e & \text{if } \epsilon = o_1, n_2 \\ 0 & \text{if } \epsilon = o_2, n_1, n_3, n_4 \end{cases} \quad \epsilon' = \begin{cases} \epsilon & \text{if } \epsilon = o_1, o_2, n_1, n_2, n_4 \\ n_4 & \text{if } \epsilon = n_3 \end{cases}$$
 and
$$G = \begin{cases} \frac{m}{2} - 1 + 2g & \text{if } \epsilon = o_1, o_2 \\ m - 2 + 2g & \text{if } \epsilon = n_1, n_2, n_3, n_4. \end{cases}$$

4.0.2. If φ maps h to 0 but maps some s_k 's to 1. Later on (Lemma 10), we shall reorder the v_j 's in the same spirit as we did for the s_k 's, and show that the isomorphism type of $\text{Ker}(\varphi)$ is in fact independent from the values of φ on the v_j 's, which reduces the computation of $\text{Ker}(\varphi)$ to the particular case where φ vanishes on all v_j 's. But before performing such a reduction, we need to show that in that particular case, $\text{Ker}(\varphi)$ is the fundamental group of a non-orientable Seifert manifold whenever N is non-orientable.

So, let us first compute $\text{Ker} \varphi$ in the particular case where φ vanishes on all v_j 's. The following lemma is an intermediate step for this computation: it gives a presentation of $\text{Ker} \varphi$ where the exceptional fibers gently appear, but where the long relation W and the \pm signs may still be of a “hybrid” form.

Lemma 4. *If φ maps s_1, \dots, s_m to 1 and all other generators to 0 then a presentation of its kernel is:*

$$\text{Ker}(\varphi) = \left\langle \begin{array}{c} s'_1, \dots, s'_{n'} \\ v'_1, \dots, v'_{g''} \\ z \end{array} \middle| \begin{array}{l} [s'_k, z] \text{ and } s'^{a'_k}_k z^{b'_k}, \quad 1 \leq k \leq n' \\ v'_j z v'^{-1}_j z^{-\epsilon'_j}, \quad 1 \leq j \leq g'' \\ z^{-2e} s'_1 \dots s'_{n'} W \end{array} \right\rangle,$$

where

- – $n' = m + 2(n - m)$,
- $(a'_k, b'_k) = (a_k/2, b_k)$ for $k \leq m$,
- $(a'_k, b'_k) = (a'_{k+n-m}, b'_{k+n-m}) = (a_k, b_k)$ for $m < k \leq n$,
- – $g'' = (m - 2) + 2g'$
- $\epsilon'_j = 1$ for $j \leq m - 2$,
- $\epsilon'_{m-2+j} = \epsilon'_{m-2+g'+j} = \epsilon_j$ for $1 \leq j \leq g'$,
- $W = \begin{cases} [v'_1, v'_2] \dots [v'_{g''-1}, v'_{g''}] & \text{if } \epsilon = o_1, o_2, \\ [v'_1, v'_2] \dots [v'_{m-3}, v'_{m-2}] v'^2_{m-1} \dots v'^2_{g''} & \text{if } \epsilon = n_1, n_2, n_3, n_4. \end{cases}$

Proof. Choosing $q = s_1$, Reidemeister-Schreier's algorithm produces a presentation of $\text{Ker}(\varphi)$ with

- generators:
 - for $1 \leq k \leq n$, $(y_k, y'_k) = \begin{cases} (s_k q^{-1}, q s_k) & \text{if } k \leq m \\ (s_k, q s_k q^{-1}) & \text{if } k > m \end{cases}$
 - for $1 \leq j \leq g'$, $(x_j, x'_j) = (v_j, q v_j q^{-1})$

- $(z, z') = (h, qhq^{-1})$.
- relations:
 - $y_1 = 1$ and $z' = z$
 - $\forall k = 1, \dots, n$,
 - * $[y_k, z] = [y'_k, z] = 1$
 - * $(y_k y'_k)^{a_k/2} z^{b_k} = 1$ if $k \leq m$
 - * $y_k^{a_k} z^{b_k} = y'^{a_k}_k z^{b_k} = 1$ if $k > m$
 - $x_j z x_j^{-1} z^{-\varepsilon_j} = x'_j z x'^{-1}_j z^{-\varepsilon_j} = 1$ ($\forall j = 1, \dots, g'$)
 - (I) $y'_2 BCX = z^e$, where $B = y_3 y'_4 y_5 y'_6 \dots y_{m-1} y'_m$, $C = y_{m+1} \dots y_n$ and

$$X = \begin{cases} [x_1, x_2] \dots [x_{2g-1}, x_{2g}] & \text{if } \epsilon = o_1, o_2 \\ x_1^2 \dots x_g^2 & \text{if } \epsilon = n_1, n_2, n_3, n_4 \end{cases}$$

- (II) $y'_1 y_2 y'_3 y_4 \dots y'_{m-1} y_m C' X' = z^e$, where $C' = y'_{m+1} \dots y'_n$ and

$$X' = \begin{cases} [x'_1, x'_2] \dots [x'_{2g-1}, x'_{2g}] & \text{if } \epsilon = o_1, o_2 \\ x'^2_1 \dots x'^2_g & \text{if } \epsilon = n_1, n_2, n_3, n_4. \end{cases}$$

(Note that X and X' allways commute with z .) Let us make a change of generators in this presentation by suppressing $y'_1, y_2, y'_3, y_4, \dots, y'_{m-1}, y_m$ and introducing instead new generators s'_1, \dots, s'_m defined by:

$$s'_k = \begin{cases} y_2'^{-1} y_3^{-1} y_4'^{-1} y_5^{-1} \dots y_{k-1}'^{-1} (y'_k y_k) y'_{k-1} \dots y_5 y'_4 y_3 y'_2 & \text{if } k \text{ is odd} \\ y_2'^{-1} y_3^{-1} y_4'^{-1} y_5^{-1} \dots y_{k-1}'^{-1} (y_k y'_k) y_{k-1} \dots y_5 y'_4 y_3 y'_2 & \text{if } k \text{ is even} \end{cases}$$

(in particular, $s'_1 = y'_1$ and $s'_2 = y_2 y'_2$). These new generators still commute with z , the relations $(y_k y'_k)^{a_k/2} z^{b_k} = 1$ become $s'^{a_k/2}_k z^{b_k} = 1$, and the relation (II) becomes (III): $A y_2'^{-1} B' C' X' = z^e$, where $A = s'_1 \dots s'_m$ and $B' = y_3^{-1} y_4'^{-1} y_5^{-1} y_6'^{-1} \dots y_{m-1}^{-1} y_m'^{-1}$. Using (I) to eliminate the generator y'_2 from (III), which will become (IV) below, we are left with the following new presentation of $\text{Ker}(\varphi)$:

- generators:
 - s'_k for $1 \leq k \leq m$
 - $y_3, y'_4, y_5, y'_6, \dots, y_{m-1}, y'_m$
 - y_k, y'_k for $m < k \leq n$
 - x_j, x'_j for $1 \leq j \leq g'$
 - z
- relations:
 - $[s'_k, z] = 1$ and $s'^{a_k/2}_k z^{b_k} = 1$ for $1 \leq k \leq m$
 - $[y_k, z] = [y'_k, z] = 1$ and $y_k^{a_k} z^{b_k} = y'^{a_k}_k z^{b_k}$ for $m < k \leq n$
 - $[y_3, z] = \dots = [y'_m, z] = 1$
 - $x_j z x_j^{-1} z^{-\varepsilon_j} = x'_j z x'^{-1}_j z^{-\varepsilon_j} = 1$ for $1 \leq j \leq g'$
 - (IV) $ABCX B' C' X' = z^{2e}$.

This last relation (IV) may be reordered by a new change of generators (replacing some of the generators by conjugates thereof) so as to become $ACC' BXB'X' = z^{2e}$, which we rewrite $ACC'(BXB'X^{-1})XX' = z^{2e}$. The parenthesis $BXB'X^{-1}$ is the product of $m-1$ elements, followed by the product (in the same order) of their inverses. It can be transformed into a product of $\frac{m}{2} - 1$ commutators, by another change of generators given by Lemma 5 below.

Provided that next (classical) lemma, this concludes the proof of Lemma 4, up to a renaming of the generators. \square

Lemma 5. *Let F_{2k+1} be the free group over g_0, \dots, g_{2k} . There exist elements $h_0, \dots, h_{2k-1} \in F_{2k+1}$ such that:*

$$g_0 g_1 \dots g_{2k} g_0^{-1} g_1^{-1} \dots g_{2k}^{-1} = [h_0, h_1][h_2, h_3] \dots [h_{2k-2}, h_{2k-1}]$$

and F_{2k+1} is the free group over $h_0, \dots, h_{2k-1}, g_{2k}$.

Proof. Let $U_i = g_{2i} \dots g_{2k}$ and $V_i = g_{2i}^{-1} \dots g_{2k}^{-1}$. Then,

$$U_k V_k = 1 \quad \text{and} \quad U_i V_i = [g_{2i} g_{2i+1}, U_{i+1} g_{2i}^{-1}] U_{i+1} V_{i+1}.$$

\square

Lemma 4 produces a standard presentation of $\text{Ker}(\varphi)$ only when $\epsilon = o_1$, or when $m = 2$ and $\epsilon \neq n_4$. In other cases, the following Lemmas 6, 7 and 8 tell how to normalize both W (which must not be a mixture of commutators and squares) and the list of the ϵ'_j 's (for which the numbers of 1's and -1 's are partially prescribed).

Lemma 6. *Let F_3 be the free group over x, y, z and $\epsilon: F_3 \rightarrow \{1, -1\}$ the morphism defined by*

$$\epsilon(x) = \epsilon(y) = 1 \quad \text{and} \quad \epsilon(z) = -1.$$

There exist elements u, v, w such that

$$\epsilon(u) = \epsilon(v) = \epsilon(w) = -1 \quad \text{and} \quad [x, y]z^2 = u^2 v^2 w^2$$

and F_3 is the free group over u, v, w .

Proof. Take for instance $u = xz$, $v = (zxz)^{-1}yz$ and $w = (yz)^{-1}z^2$. \square

Lemma 7. *Let F_4 be the free group over x, y, z, t and $\epsilon: F_4 \rightarrow \{1, -1\}$ the morphism defined by*

$$\epsilon(x) = \epsilon(y) = 1 \quad \text{and} \quad \epsilon(z) = \epsilon(t) = -1.$$

There exist elements $x', y', z', t' \in F_4$ such that

$$\epsilon(x') = \epsilon(y') = \epsilon(z') = \epsilon(t') = -1 \quad \text{and} \quad [x, y][z, t] = [x', y'][z', t']$$

and F_4 is the free group over t', u', v', w' .

Proof. Take for instance

$$x' = xyz, \quad y' = z^{-1}x^{-1}, \quad z' = (y^{-1}z)^{-1}z(y^{-1}z) \quad \text{and} \quad t' = tz^{-1}(y^{-1}z).$$

\square

Lemma 8. *Let F_4 be the free group over t, u, v, w and $\epsilon: F_4 \rightarrow \{1, -1\}$ the morphism defined by*

$$\epsilon(t) = \epsilon(u) = \epsilon(v) = 1 \quad \text{and} \quad \epsilon(w) = -1.$$

There exist elements $t', u', v', w' \in F_4$ such that

$$\epsilon(t') = 1, \quad \epsilon(u') = \epsilon(v') = \epsilon(w') = -1 \quad \text{and} \quad t^2 u^2 v^2 w^2 = t'^2 u'^2 v'^2 w'^2$$

and F_4 is the free group over x', y', z', t' .

Proof. Take for instance t', u', w', v' successively defined by:

$$t' = tu^2vu^{-1}, \quad t'u' = u^2vw \quad u'w' = uvw^2 \quad \text{and} \quad u'v'w' = w.$$

\square

Using Lemmas 6, 7 and 8, we deduce from Lemma 4:

Proposition 9. *If φ maps s_1, \dots, s_m to 1 and all other generators to 0 then its kernel is the fundamental group of the Seifert manifold given by*

$$\{e'; (\epsilon', G); F_m\}$$

where

$$e' = \begin{cases} 2e & \text{if } \epsilon = o_1, n_2 \\ 0 & \text{if } \epsilon = o_2, n_1, n_3, n_4 \end{cases} \quad \epsilon' = \begin{cases} \epsilon & \text{if } \epsilon = o_1, o_2, n_1, n_2, n_4 \\ n_4 & \text{if } \epsilon = n_3 \end{cases}$$

$$G = \begin{cases} \frac{m}{2} - 1 + 2g & \text{if } \epsilon = o_1, o_2 \\ m - 2 + 2g & \text{if } \epsilon = n_1, n_2, n_3, n_4 \end{cases}$$

and F_m as defined in Notation 2.

Proof. When $\epsilon = o_1$, Lemma 4 directly gives the result and $\epsilon' = o_1$, with $2G = g'' = m - 2 + 2g' = m - 2 + 4g$. When $\epsilon = o_2$, the number of ϵ'_j 's equal to -1 is $g' = 2g > 0$ hence by Lemma 7, all ϵ'_j 's can be replaced by -1 , so $\epsilon' = o_2$ and $2G = m - 2 + 4g$ as in the previous case. When $\epsilon = n_1$, Lemma 6 may be applied if necessary (i.e. if $m > 2$) to replace each commutator by a product of two squares (only a weak form of the Lemma is used here, forgetting about the morphism ε of its statement). Thus we get $\epsilon' = n_1$ and $G = g'' = m - 2 + 2g' = m - 2 + 2g$. When $\epsilon = n_2$, applying again Lemma 6 if necessary (in its strong form) gives the result and $\epsilon' = n_2$ with $G = m - 2 + 2g$ as in the previous case. When $\epsilon = n_3$, $W = [v'_1, v'_2] \dots [v'_{m-3}, v'_{m-2}] v'^2_{m-1} \dots v'^2_{m-2+2g}$ and among $\epsilon'_{m-1}, \dots, \epsilon'_{m-2+2g}$, there are two 1's, but the number of -1 's is $2g - 2 > 0$ and we may reorder these $2g$ ϵ'_j 's to put the two 1's first (by a repeated change of variable, using that $u^2 v^2 = (u^2 v u^{-2})^2 u^2$). Lemma 6 again gives the conclusion. When $\epsilon = n_4$ the same method allows to transform W into a product of squares but we are left with four 1's instead of two. Reordering again to put these four 1's at the beginning, Lemma 8 allows to reduce this number of 1's to two, hence (again) $\epsilon' = n_4$ (and $G = m - 2 + 2g$). \square

The next lemma will be used in Theorem 11 to extend the result of the previous proposition to the general case, where φ does not necessarily vanish on all v_j 's.

Lemma 10. *If two morphisms from $\pi_1(N)$ to \mathbb{Z}_2 map s_1, \dots, s_m ($m > 0$) to 1 and h, s_{m+1}, \dots, s_n to 0 then their kernels are isomorphic.*

Proof.

When $\epsilon = n_1, n_2, n_3, n_4$, the idea is to change the presentation of $\pi_1(N)$, using $\varphi(s_1) = 1$ to “kill” all the $\varphi(v_j)$'s, one after another. The formulas are simpler if we prepare each of these “murders” by temporarily permuting the $\varphi(v_j)$'s, to put at the end the one whose value we want to switch from 1 to 0.

The values $\varphi(v_j)$ and $\varphi(v_{j+1})$ can be exchanged by the following change of generators (leaving the other generators untouched):

$$v'_j = v_j^2 v_{j+1} v_j^{-2}, v'_{j+1} = v_j.$$

Combining such transpositions, we can temporarily reorder the v_j 's (it may affect the conventional ordering of the ε_j 's when $\epsilon = n_3, n_4$, but this is temporary hence harmless).

Then, the value of $\varphi(v_g)$ can be switched from 1 to 0 by:

$$v'_g = v_g s_1, \quad s'_1 = v_g^{-1} s_1^{-1} v'_g.$$

When v_g anticommutes with h , this is in fact (like the previous one) an automorphism of $\pi_1(N)$, but when v_g commutes with h , it is only a change of presentation, since b_1 is changed into its opposite. Thus, after “killing” all $\varphi(v_j)$'s and then applying Proposition 9, we get the same invariants for $\text{Ker}(\varphi)$ as if all $\varphi(v_j)$'s were already 0, up to a possible change of sign of b_1 in the type $(a_1/2, b_1)$ of the first exceptional fiber. However, this may only happen when some ε_j 's are equal to 1, i.e. when $\epsilon = n_1, n_3, n_4$. But we see from Proposition 9 that $\text{Ker}(\varphi)$ inherits this non-orientability, which allows to replace in the final result such an $(a_1/2, -b_1)$ (if it occurs) by $(a_1/2, b_1)$.

When $\epsilon = o_1, o_2$, the idea is the same: the value of $(\varphi(v_{2i-1}), \varphi(v_{2i}))$ can be exchanged with that of $(\varphi(v_{2i+1}), \varphi(v_{2i+2}))$ by:

$$\begin{aligned} v'_{2i-1} &= [v_{2i-1}, v_{2i}]v_{2i+1}[v_{2i-1}, v_{2i}]^{-1}, & v'_{2i} &= [v_{2i-1}, v_{2i}]v_{2i+2}[v_{2i+2}, v_{2i}]^{-1}, \\ v'_{2i+1} &= v_{2i-1}, & v'_{2i+2} &= v_{2i}, \end{aligned}$$

and the values of $\varphi(v_j)$ and $\varphi(v_{j+1})$ when j is odd can be exchanged by:

$$v'_j = v_j v_{j+1} v_j^{-1}, \quad v'_{j+1} = v_j^{-1}.$$

The value of $\varphi(v_{2g})$ can be switched from 1 to 0 by:

$$s'_1 = v_{2g-1} s_1 v_{2g-1}^{-1}, \quad v'_{2g-1} = s'_1 v_{2g-1} s'^{-1}_1, \quad v'_{2g} = s'_1 s'^{-1}_1 v_{2g} s'^{-1}_1.$$

When $\epsilon = o_2$, this again changes b_1 to $-b_1$ but it can be cured if necessary in the final result, by the same argument. □

From Proposition 9 and Lemma 10 we immediately deduce:

Theorem 11. *If φ map s_1, \dots, s_m ($m > 0$) to 1 and h, s_{m+1}, \dots, s_n to 0 then its kernel is the fundamental group of the Seifert manifold given by*

$$\{e'; (\epsilon', G); F_m\}$$

where

$$e' = \begin{cases} 2e & \text{if } \epsilon = o_1, n_2 \\ 0 & \text{if } \epsilon = o_2, n_1, n_3, n_4 \end{cases} \quad \epsilon' = \begin{cases} \epsilon & \text{if } \epsilon = o_1, o_2, n_1, n_2, n_4 \\ n_4 & \text{if } \epsilon = n_3 \end{cases}$$

$$G = \begin{cases} \frac{m}{2} - 1 + 2g & \text{if } \epsilon = o_1, o_2 \\ m - 2 + 2g & \text{if } \epsilon = n_1, n_2, n_3, n_4 \end{cases}$$

and F_m as defined in Notation 2.

4.0.3. *If φ maps some v_j 's to 1 and all other generators 0. In this subsection, $\varphi(h) = \varphi(s_1) = \dots = \varphi(s_n) = 0$. Apart from orientation covers and some “exotic” cases, $\text{Ker}(\varphi)$ will have the same description as in Theorem 11, with m replaced by 0. The two cases $\epsilon = o_1, o_2$ and $\epsilon = n_1, n_2, n_3, n_4$ will be treated separately (Propositions 12 and 14) and the global result will be rephrased in Theorem 15.*

Proposition 12. *When φ maps $r > 0$ generators v_j 's to 1 and all other generators to 0 and $\epsilon = o_1$ or o_2 , $\text{Ker}(\varphi)$ is the fundamental group of the Seifert manifold given by the following invariants, with F_{OC} and F_0 as defined in Notation 2.*

- if $\epsilon = o_1$: $\{2e; (o_1, 2g - 1); F_0\}$
- if $\epsilon = o_2$ and $r = g$ (orientation cover): $\{0; (o_1, 2g - 1); F_{OC}\}$

- in all other cases of $\epsilon = o_2$: $\{0; (o_2, 2g - 1); F_0\}$.

Proof. Let us reorder r the number of v_j 's mapped to 1 in such a way that $\varphi(v_j) = 1 \Leftrightarrow j \leq r$ (by the same method as in Lemma 10). Let ε be the common value of the ε_j 's, i.e. $\varepsilon = 1$ if $\epsilon = o_1$ and to $\varepsilon = -1$ if $\epsilon = o_2$.

Choosing $q = v_r$, Reidemeister-Schreier's algorithm produces a presentation of $\text{Ker}(\varphi)$ with

- generators:

- for $1 \leq k \leq n$, $(y_k, y'_k) = (s_k, qs_kq^{-1})$
- for $1 \leq j \leq 2g$,

$$(x_j, x'_j) = \begin{cases} (v_jq^{-1}, qv_j) & \text{if } j \leq r \\ (v_j, qv_jq^{-1}) & \text{if } j > r \end{cases}$$

- $(z, z') = (h, qhq^{-1})$

- relations:

- $x_r = 1$, $z' = z^\varepsilon$
- z commutes with all y_k 's and y'_k 's, and with x_j and x'_j for $j \leq r$
- $x_jzx_j^{-1}z^{-\varepsilon} = x'_jzx_j'^{-1}z^{-\varepsilon} = 1$ for $j > r$
- $y_k^{a_k}z^{b_k} = y_k'^{a_k}z'^{b_k} = 1$ ($\forall k = 1, \dots, n$)
- (I) $YB = z^e$, (II) $Y'B' = z^{\varepsilon re}$, where $Y = y_1 \dots y_n$, $Y' = y'_1 \dots y'_n$, and B, B' are described as follows:

- if r is odd, $B = ZX'_{r+1}x_{r+1}^{-1}X$ and $B' = Z'x'_rx_{r+1}x_r'^{-1}x_{r+1}'^{-1}X'$, where

$$X = [x_{r+2}, x_{r+3}] \dots [x_{2g-1}, x_{2g}], \quad X' = [x'_{r+2}, x'_{r+3}] \dots [x'_{2g-1}, x'_{2g}],$$

$$Z = (x_1x'_2x_1'^{-1}x_2^{-1}) \dots (x_{r-2}x'_{r-1}x_{r-2}'^{-1}x_{r-1}^{-1}),$$

$$Z' = (x'_1x_2x_1^{-1}x_2'^{-1}) \dots (x'_{r-2}x_{r-1}x_{r-2}^{-1}x_{r-1}'^{-1});$$

- if r is even, $B = ZX_{r-1}x'_rx_{r-1}'^{-1}X$ and $B' = Z'x'_{r-1}x_{r-1}^{-1}x'_r'^{-1}X'$, where

$$X = [x_{r+1}, x_{r+2}] \dots [x_{2g-1}, x_{2g}], \quad X' = [x'_{r+1}, x'_{r+2}] \dots [x'_{2g-1}, x'_{2g}],$$

$$Z = (x_1x'_2x_1'^{-1}x_2^{-1}) \dots (x_{r-3}x'_{r-2}x_{r-3}'^{-1}x_{r-2}^{-1}),$$

$$Z' = (x'_1x_2x_1^{-1}x_2'^{-1}) \dots (x'_{r-3}x_{r-2}x_{r-3}^{-1}x_{r-2}'^{-1}).$$

Assume first that r is odd. Using (II) to eliminate x'_{r+1} in (I), and replacing some of the generators by conjugates thereof (without altering the previous relations) to reorder the subexpressions of the resulting relation, (I) becomes:

$$YY'W = z^{(1+\varepsilon)e}, \quad \text{with} \quad W = ZZ'[x'_r, x_{r+1}]X'X.$$

Transforming ZZ' by lemma 13 below (which is stated informally but whose proof gives explicit formulas), W becomes a product of $(r-1)+1+(2g-r-1)$ commutators of new generators, whose associated ε'_j 's are equal to 1 for the first $2(r-1)+1$ of them and to ε for the $1+2(2g-r-1)$ last ones. We thus get the following Seifert invariants for $\text{Ker}(\varphi)$:

- if $\epsilon = o_1$: $\{2e; (o_1, 2g - 1); F_0\}$
- if $\epsilon = o_2$ (noting that $1 + 2(2g - r - 1) > 0$): $\{0; (o_2, 2g - 1); F_{OC}\}$.

In the o_2 (non-orientable) case, all $-b_k$'s can be replaced by their opposites, i.e. F_{OC} replaced by F_0 , which concludes the odd case.

Assume now that r is even. Using (I) to eliminate x'_{r-1} and performing similar transformations, (II) becomes:

$$Y'YW = z^{(1+\varepsilon)e}, \quad \text{with} \quad W = Z'Z[x_{r-1}, x'_r]XX'.$$

Using lemma 13 again, W becomes a product $(r-2) + 1 + (2g-r)$ commutators and the first $2(r-2) + 2\varepsilon_j$'s are equal to 1, the $2(2g-r)$ last ones being equal to ε . Hence the conclusion is the same as in the odd case, except when $\varepsilon = -1$ and $2(2g-r) = 0$, which corresponds to the orientation cover case of the statement. This concludes the proof of Proposition 12, provided the next Lemma. \square

Lemma 13. *In any group, an expression of the form*

$$(a_1b_1c_1d_1) \dots (a_kb_kc_kd_k)(c_1^{-1}d_1^{-1}a_1^{-1}b_1^{-1}) \dots (c_k^{-1}d_k^{-1}a_k^{-1}b_k^{-1})$$

is the product of $2k$ commutators.

Proof. Let $U_k = (a_1b_1c_1d_1) \dots (a_kb_kc_kd_k)$ and $V_k = (c_1^{-1}d_1^{-1}a_1^{-1}b_1^{-1}) \dots (c_k^{-1}d_k^{-1}a_k^{-1}b_k^{-1})$. Then $U_0V_0 = 1$, and $U_{k+1}V_{k+1}$ is the product of U_kV_k (which by induction hypothesis is a product of $2k$ commutators) by

$$V_k^{-1}a_{k+1}b_{k+1}c_{k+1}d_{k+1}V_k c_{k+1}^{-1}d_{k+1}^{-1}a_{k+1}^{-1}b_{k+1}^{-1} = \\ [V_k^{-1}a_{k+1}, b_{k+1}V_k](b_{k+1}a_{k+1})[V_k^{-1}c_{k+1}, d_{k+1}V_k](b_{k+1}a_{k+1})^{-1}.$$

\square

Proposition 12 dealt with the case $\epsilon = o_1, o_2$. The next proposition deals with the other case, $\epsilon = n_1, n_2, n_3, n_4$.

Proposition 14. *When φ maps some v_j 's to 1 and all other generators to 0 and $\epsilon = n_1, n_2, n_3$ or n_4 , $\text{Ker}(\varphi)$ is the fundamental group of the Seifert manifold given by the following invariants, with F_{OC} and F_0 as defined in Notation 2.*

- (Orientation covers)
 - if $\epsilon = n_1$ and φ maps all v_j 's to 1: $\{0; (o_1, g-1); F_{OC}\}$
 - if $\epsilon = n_3$ and φ sends only v_1 to 1, or if $\epsilon = n_4$ and φ sends only v_1, v_2 to 1: $\{0; (n_2, 2g-2); F_{OC}\}$
- (Exotic cases) if φ maps all v_j 's to 1 but $\epsilon \neq n_1$
 - if $\epsilon = n_2$: $\{2e; (o_1, g-1); F_0\}$
 - if $\epsilon = n_3, n_4$: $\{0; (o_2, g-1); F_0\}$
- (Ordinary cases) in all other cases: $\{\epsilon'; (\epsilon', 2g-2); F_0\}$ with

$$\epsilon' = \begin{cases} \epsilon & \text{if } \epsilon = n_1, n_2, n_4 \\ n_4 & \text{if } \epsilon = n_3 \end{cases} \quad \text{and} \quad e' = \begin{cases} 2e & \text{if } \epsilon = n_2 \\ 0 & \text{if } \epsilon = n_1, n_3, n_4. \end{cases}$$

Proof. Assume, like in the proof of Proposition 12, that $\varphi(v_j) = 1 \Leftrightarrow j \leq r$. We shall have to be cautious about a new phenomenon: this reordering of the v_j 's may affect the ordering of the ε_j 's, i.e. we shall try to maintain the convention that the ε_j 's equal to 1 are allways ε_1 when $\epsilon = n_3$ and $\varepsilon_1, \varepsilon_2$ when $\epsilon = n_4$, but we shall locally drop this convention inside the present proof whenever it is not compatible with our reordering of the v_j 's.

Choosing $q = v_1$, Reidemeister-Schreier's algorithm produces a presentation of $\text{Ker}(\varphi)$ with

- generators:

- for $1 \leq k \leq n$, $(y_k, y'_k) = (s_k, qs_kq^{-1})$
- for $1 \leq j \leq g$,

$$(x_j, x'_j) = \begin{cases} (v_jq^{-1}, qv_j) & \text{if } j \leq r \\ (v_j, qv_jq^{-1}) & \text{if } j > r \end{cases}$$

- $(z, z') = (h, qhq^{-1})$

- relations:

- $x_1 = 1$, $z' = z^{\varepsilon_1}$
- z commutes with all y_k 's and y'_k 's
- $x_jzx_j^{-1}z^{-\varepsilon'_j} = x'_jzx'_j{}^{-1}z^{-\varepsilon'_j} = 1$ whith $\varepsilon'_j = \begin{cases} \varepsilon_j\varepsilon_1 & \text{if } j \leq r \\ \varepsilon_j & \text{if } j > r \end{cases}$
- $y_k^{a_k}z^{b_k} = y'_k{}^{a_k}z^{\varepsilon_1b_k} = 1$ ($\forall k = 1, \dots, n$)
- (I) $Yx'_1ZX = z^e$, where $Y = y_1 \dots y_n$, $X = x_{r+1}^2 \dots x_n^2$ and $Z = x_2x'_2 \dots x_rx'_r$
- (II) $Y'x'_1Z'X' = z^{\varepsilon_1e}$, where $Y' = y'_1 \dots y'_n$, $X' = x'_{r+1} \dots x'_n$ and $Z' = x'_2x_2 \dots x'_rx_r$.

Eliminating x'_1 , (I) and (II) join to become (III): $YY'^{-1}X'^{-1}Z'^{-1}ZX = z^{(1-\varepsilon_1)e}$ and $Z'^{-1}Z$ is a product of $r-1$ commutators (of conjugates of inverses of $x_2, x'_2, \dots, x_r, x'_r$, having the same ε'_j 's).

When $r = g$, X and X' are empty products hence $\epsilon' = o_1$ or o_2 . Moreover when $\epsilon = n_3, n_4$, since we find $\epsilon' = o_2$, we can replace the $-b_k$'s which occur by their opposites. More precisely Seifert invariants for $\text{Ker}(\varphi)$ when $r = g$ are:

- if $\epsilon = n_1$: $\{0; (o_1, g-1); F_{OC}\}$
- if $\epsilon = n_2$: $\{2e; (o_1, g-1); F_0\}$
- if $\epsilon = n_3, n_4$: $\{0; (o_2, g-1); F_0\}$.

When $r < g$, (III) contains a product of $2(g-r)$ squares and $r-1$ commutators, which can be converted to a product of $2g-2$ squares (using Lemma 6 and taking care of the ε'_j 's to determine ϵ'). Moreover, when the ϵ' we find corresponds to a non-orientable manifold, all $b'_k = -b_k$'s (if any) can be replaced by $b'_k = b_k$. Hence Seifert invariants for $\text{Ker}(\varphi)$ when $r < g$ are:

- if $\epsilon = n_2$: $\{2e; n_2, 2g-2; F_0\}$
- if $\epsilon = n_3$ and φ sends only v_1 to 1, or if $\epsilon = n_4$ and φ sends only v_1, v_2 to 1: $\{0; (n_2, 2g-2); F_{OC}\}$
- in all other cases: $\{0; \epsilon', 2g-2; F_0\}$ with $\epsilon' = n_1$ if $\epsilon = n_1$, and $\epsilon' = n_4$ if $\epsilon = n_3, n_4$.

□

The following theorem is a synthesis of Propositions 12 and 14.

Theorem 15. *When φ maps some v_j 's to 1 and all other generators to 0, $\text{Ker}(\varphi)$ is the fundamental group of the Seifert manifold given by the following invariants, whith F_{OC} and F_0 as defined in Notation 2.*

- (Orientation covers)
 - if $\epsilon = o_2$ and φ maps all v_j 's to 1: $\{0; (o_1, 2g-1); F_{OC}\}$
 - if $\epsilon = n_1$ and φ maps all v_j 's to 1: $\{0; (o_1, g-1); F_{OC}\}$

- if $\epsilon = n_3$ and φ sends only v_1 to 1, or if $\epsilon = n_4$ and φ sends only v_1, v_2 to 1: $\{0; (n_2, 2g - 2); F_{OC}\}$
- (Exotic cases) if $\epsilon = n_2, n_3, n_4$ and φ maps all v_j 's to 1:
 - if $\epsilon = n_2$: $\{2e; (o_1, g - 1); F_0\}$
 - if $\epsilon = n_3, n_4$: $\{0; (o_2, g - 1); F_0\}$
- (Ordinary cases) in all other cases: $\{e'; (\epsilon', G); F_0\}$ with

$$e' = \begin{cases} 2e & \text{if } \epsilon = o_1, n_2 \\ 0 & \text{if } \epsilon = o_2, n_1, n_3, n_4 \end{cases} \quad \epsilon' = \begin{cases} \epsilon & \text{if } \epsilon = o_1, o_2, n_1, n_2, n_4 \\ n_4 & \text{if } \epsilon = n_3 \end{cases}$$
 and $G = \begin{cases} 2g - 1 & \text{if } \epsilon = o_1, o_2 \\ 2g - 2 & \text{if } \epsilon = n_1, n_2, n_3, n_4. \end{cases}$

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Anne Bauval

Institut de Mathématiques de Toulouse
 Equipe Emile Picard, UMR 5580
 Université Toulouse III
 118 Route de Narbonne, 31400 Toulouse - France
 e-mail: bauval@math.univ-toulouse.fr

Claude Hayat

Institut de Mathématiques de Toulouse
 Equipe Emile Picard, UMR 5580
 118 Route de Narbonne, 31400 Toulouse - France
 e-mail: hayat@math.univ-toulouse.fr